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Regularization of Minimum Parameter Attitude Estimation

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Introduction

THE mathematical description of the main directional properties of a satellite body can be achieved by unit vectors in three- and four-dimensional Euclidean space. A three-dimensional unit vector corresponds to the spin-axis direction of a spin-stabilized satellite. In this case, the minimum number of parameters to define a unit vector uniquely is two and this corresponds to a point on the S^2 sphere. The four well-known Euler–Rodrigues parameters, together forming a unit vector, represent a rotation. These parameters were first derived by Euler [1] in 1771 as a rational parameterization of an orthogonal transformation for which the determinant is equal to 1 and later independently rediscovered by Rodrigues [2] in 1840 in the context of the composition rule of three-dimensional rotations. Although the composition rule can be understood as a quaternion multiplication, we will only use the aforementioned scalar parameters in what we call the *Rodrigues four-vector* [3]. The sphere in four dimensions is thus a locus for these vectors, which are able to represent any instantaneous three-axis attitude of a non-spin-stabilized satellite. The minimum number of parameters uniquely defining a Rodrigues four-vector is three.

Unfortunately, all of the minimum parameterizations on S^2 and S^3 are known to become undefined at certain points. In the context of estimation, these points have been considered to be unsurmountable singularities until recently. Such a singularity occurring with Euler angles is often called a *gimbal lock*. We will show that by rearranging a differential correction to become a *geodetic correction*, the problem partly disappears. To achieve this, we do not determine the differentials of the parameters themselves, but rather the components of a differential displacement or arc length. This move is only helpful for particular parameterizations, as we will show in the next section. Nevertheless, even then, some of the parameters themselves are still undetermined at the singularities; consequently, the differential update of these parameters is undefined as well. This is a second problem to solve, for we need to update the parameters in each correction step. In practice, depending on the type of parameterization, two different situations occur. Either we can approach the original singularity without problems, except when extremely close to it or, alternatively, convergence of a steepest descent is already hampered further away from the singularity. In the latter case, we have shown that this can be solved by what we have called a *loxodromic descent* [4]. In the former case, when close enough to the singularity, the actual value of the undetermined parameter has no significance anymore and the corresponding differential update can, for instance, arbitrarily be replaced by zero. This Note is devoted to the case that always applies on S^2 and relies on such parameterizations of the Rodrigues four-vector, which can dispense with a loxodromic descent on S^3.

The application of a geodetic correction method to estimate a location on a known curved surface is a new technique that can be applied to other areas as well. It guarantees that the step length of the correction is the optimal estimate of an Euclidean geodetic differential step. Thereby, the efficiency of estimation should become more or less independent of the local shape, if not close to a saddle point or discontinuity.

Metric or Algebraic Regularization

To understand the basic principles behind the proposed algebraic regularization, we will make a brief excursion into differential geometry. To this aim, we assume that $(x_1, \ldots, x_k)$ are orthogonal Cartesian coordinates. We now transform these coordinates into curvilinear coordinates $(y_1, \ldots, y_\ell)$, subject to the transformation

$$x_1 = f_1(y_1, \ldots, y_\ell) \qquad \cdots \qquad x_k = f_k(y_1, \ldots, y_\ell) \tag{1}$$

where $y_1, \ldots, y_\ell$ can be supposed to be mutually independent if $k \geq \ell$ and will, in our case, define a sphere with the minimum number of parameters: namely, $\ell = k - 1$ for $3 \leq k$. The differentials of the Cartesian coordinates are mutually orthogonal infinitesimal line elements equal to

$$dx_j = \sum_{i=1}^{\ell} \frac{\partial f_j}{\partial y_i} dy_i \quad (j = 1, \ldots, k)$$

Working in Euclidean space, the square of the total resulting infinitesimal length $ds^2 = dx_1^2 + \cdots + dx_k^2$ bound to the surface defined by Eq. (1) must remain the same in the transformed system. Thus,

$$ds^2 = \sum_{m=1}^{k} dx_m^2 = \sum_{i}^{\ell} \sum_{j}^{\ell} g_{ij} dy_i dy_j \tag{2}$$

with

$$g_{ij} = \sum_{m=1}^{k} \frac{\partial f_m}{\partial y_i} \frac{\partial f_m}{\partial y_j} \tag{3}$$

must apply, where g_{ij} is a well-known *metrical coefficient* of the curvilinear surface [5] defined by Eq. (1). If the infinitesimal parameters dy_i are locally orthogonal to the other components dy_j with $j \neq i$ everywhere except in an area of measure zero, then all g_{ij} must vanish identically. In this case, we show that such a line element along the local coordinate line y_i has the infinitesimal length $du_i = \sqrt{g_{ii}} dy_i$, and we will exploit the fact that this length is always defined.

To demonstrate this assertion, we first translate Eq. (2) into a matrix equation involving the vectors $\mathbf{dx}' = |dx_1, \ldots, dx_k|$ and $\mathbf{dy}' = |dy_1, \ldots, dy_\ell|$, where the prime denotes transposition. This

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leads to $\mathbf{dx} = B\mathbf{dy}$, where B is a $k \times \ell$ matrix with the elements $b_{ij} = \partial f_i/\partial y_j$. The condition that $g_{ij} = 0$ for $i \neq j$ in Eq. (3) is then equivalent to

$$0 = g_{ij} = |b_{1i}, b_{2i}, \ldots, b_{ki}||b_{1j}, b_{2j}, \ldots, b_{kj}|' \quad (i \neq j)$$

This means that the ℓ columns of the B matrix must be mutually orthogonal, but they are not necessarily normalized, because

$$g_{ii} = \|b_{1i}, b_{2i}, \ldots, b_{ki}\|^2 \tag{4}$$

is often not equal to one and may even become zero at singular locations. To achieve normalization, we introduce the ℓ-dimensional diagonal matrix D (for which the ith diagonal element is equal to $\sqrt{g_{ii}}$) and observe that by setting $B = CD$, we find that C is an incomplete orthonormal matrix missing $k - \ell$ columns to be complete. Consequently, C must be a full-rank matrix because $C'C = I_\ell$ must always hold true. The matrix I_ℓ is the ℓ-dimensional unit matrix. If we now write

$$\mathbf{dx} = C(D\mathbf{dy}) = C\mathbf{du} \tag{5}$$

we see that by replacing the estimation of y_i by the estimation of $u_i = \sqrt{g_{ii}}y_i$, a differential correction step will never be singular because C has always full rank. Instead of dy_i being undefined at a given location, we will have $du_i = 0$ at that point and, in this way, achieve an algebraic regularization. It is important to note that Eq. (5) is a locally defined *one-to-one* infinitesimal transformation. Consequently, if $\mathbf{dx}$ and $\mathbf{du}$ are replaced by small random values $\boldsymbol{\Delta}\mathbf{x}$ and $\boldsymbol{\Delta}\mathbf{u}$ subject to Eq. (5) occurring in an estimation, the orthogonality of C leads to

$$\begin{aligned} \overline{\boldsymbol{\Delta} s^2} &= E(\boldsymbol{\Delta}\mathbf{x}'\boldsymbol{\Delta}\mathbf{x}) = E(\boldsymbol{\Delta}\mathbf{u}'\boldsymbol{\Delta}\mathbf{u}) \\ Q &= E(\boldsymbol{\Delta}\mathbf{x}\boldsymbol{\Delta}\mathbf{x}') = CE(\boldsymbol{\Delta}\mathbf{u}\boldsymbol{\Delta}\mathbf{u}')C' \end{aligned} \tag{6}$$

where $\varepsilon = \overline{\Delta s^2}$ is the mean error and Q is the k-dimensional covariance matrix of $\boldsymbol{\Delta}\mathbf{x}$ subject to the constraint to satisfy Eq. (1) for any real values of y_i. We propose to call a differential correction based on this principle a *geodetic correction*, and to C we give the name *plug-in matrix*. Note that this way of estimating a differential update does not, in principle, happen in a tangential plane. It thereby avoids the problem of reducing a nonminimal number of components (for instance, the x_i), first to the tangential plane and from that plane further to the curvilinear surface, a procedure that normally implies assumptions that are suboptimal but, in the favorable case, consistent in the limit of a convergent steepest descent.

A useful byproduct of Eq. (6) in dimension four is the fact that the kinematical equations linking the partial time derivative of the Rodrigues four-vector $\mathbf{x}$ to the partial time derivatives of the geodetic vector $\mathbf{u}$ on S^3 are simply

$$\dot{\mathbf{x}} = C\dot{\mathbf{u}} \quad \text{and} \quad C'\dot{\mathbf{x}} = \dot{\mathbf{u}} \tag{7}$$

All types of conventional spherical coordinates parametrizing a sphere of any finite dimension are orthogonal parameterizations, which can thus be algebraically regularized by estimating geodetic corrections. Thereby, the original singularity (namely, the indetermination of certain values of y_i at certain locations) will reappear in a more tractable way. We also need to update the values of all y_i at each correction step to recompute the derivatives of the functions in Eq. (1) and obtain new values for the metric coefficients even when some g_{ii} get very close to zero. Thus, we have a one-to-one nonsingular transformation from the k components of $\mathbf{dx}$ to the $k-1$ components of $\mathbf{du}$. From the latter, we then further need to determine updates of the $k-1$ coordinates of $\mathbf{y}$ that are not one-to-one in an area of measure zero, corresponding to the original singularity. Using 64-bit floating-point arithmetic, we will propose the adequate transformations in which the small values of g_{ii} can be handled exactly up to absolute values equal to 10^{-12}, a value beyond which updates of the corresponding y_i are no longer considered to be *unique* and are selected ad hoc. In the problem at hand, we will rely on two further insights to find these algorithms. First, we will see that there is always at least one parameter y_i that remains unaffected by such a problem. Second, we can reasonably trust the value of the total magnitude of the geodetic correction, because the algebra to obtain it is regular.

Basic Parameterizations and Geodetic Corrections

Parametrizing the Sphere in Three Dimensions

All conventional spherical coordinates defining points on S^2 are polar coordinates (they have an angle directly related to a *pole*, independently of the other spherical coordinates) and can be reduced to the concept of a right ascension α and a declination δ. These angles parametrize the unit three-vector as follows:

$$x_1 = \cos\alpha\cos\delta \qquad x_2 = \sin\alpha\cos\delta \qquad x_3 = \sin\delta \tag{8}$$

Computing the infinitesimal line (arc) element, as prescribed in Eq. (3), yields

$$ds^2 = \cos^2\delta d\alpha^2 + d\delta^2$$

The orthogonal metric line elements are thus $d\delta$ and $\cos\delta d\alpha$; consequently,

$$\begin{vmatrix} dx_1 \\ dx_2 \\ dx_3 \end{vmatrix} = \begin{vmatrix} -\sin\alpha & -\cos\alpha\sin\delta \\ \cos\alpha & -\sin\alpha\sin\delta \\ 0 & \cos\delta \end{vmatrix} \begin{vmatrix} \cos\delta d\alpha \\ d\delta \end{vmatrix} = C\mathbf{du} \tag{9}$$

By inspection, we see, as shown before, that the 3×2 matrix C is always of rank 2 for any values of α and δ. Thus, if we have to estimate $\mathbf{x}$ starting from an n-dimensional vector function $\mathbf{e}(\mathbf{x})$ being equal to a measurement vector $\mathbf{m}$ of the same dimension or $\mathbf{e}(\mathbf{x}) = \mathbf{m}$, a differential correction starts from the linearization:

$$\begin{aligned} \mathbf{e}(\mathbf{x_0}) + \left(\frac{\partial \mathbf{e}(\mathbf{x})}{\partial \mathbf{x}}\right)_0 \boldsymbol{\Delta}\mathbf{x} &= \mathbf{e}(\mathbf{x_0}) + A\boldsymbol{\Delta}\mathbf{x} \\ &= \mathbf{e}(\alpha_0, \delta_0) + (AC_0)\boldsymbol{\Delta}\mathbf{u} = \mathbf{m} \end{aligned} \tag{10}$$

where the subscript 0 is always employed to denote an initial condition in a linearization/correction step. The last equation inside Eq. (10) represents the regularized minimum parameter linearization, which is then adapted to obtain an optimal estimate. One could object at this point that if $\mathbf{e}(\mathbf{x})$ was linear at the outset, it becomes nonlinear by introducing $\mathbf{u}$ and including the matrix C in the penultimate equation member. The reply is simple. If we stuck to $\boldsymbol{\Delta}\mathbf{x}$ instead of $\boldsymbol{\Delta}\mathbf{u}$, we would have to include the normality constraint applied to $\mathbf{x}$ and in a batch estimation we will, in the end, reach an estimation of exactly the same quality. In a sequential estimation, however, the normality constraint is optimally taken care of by the method proposed, whereas the step-by-step application of Lagrange multipliers along the sequence, for instance, will be suboptimal.

Assume now that we have obtained a geodetic correction $\boldsymbol{\Delta}\mathbf{u}$. How do we proceed? We observe that the declination always stays regular, whereas the right ascension is undefined if $\cos\delta = 0$. Therefore, we compute the improved $\delta = \delta_0 + \Delta u_2$ first. If the updated value of $|\cos\delta| < \epsilon$, we set $\Delta u_1 = 0$; otherwise, we compute $\Delta\alpha = \Delta u_1/\cos\delta$. A last protection against a numerical runaway consists of systematically applying modulo 2π to α. The value of ϵ depends on the floating-point precision with which one is working. For 64-bit floating-point representations, we propose to go to $\epsilon = 10^{-12}$.

Parametrizing the Sphere in Four Dimensions

We start by giving two examples in which the sphere is parametrized by three nonspherical angles: namely, the Euler angles. Let us start with the conventional (symmetric) Euler angles, implying that the first rotation over the angle ϕ is performed around the x_3 axis, the second over an angle θ around x_1, and the third over an angle ψ, again around x_3. The parameterization of the Rodrigues four-vector by means of these Euler angles is

$$x_1 = \cos[0.5(\phi - \psi)] \sin 0.5\theta \qquad x_2 = \sin[0.5(\phi - \psi)] \sin 0.5\theta$$
$$x_3 = \sin[0.5(\psi + \phi)] \cos 0.5\theta \qquad x_4 = \cos[0.5(\psi + \phi)] \cos 0.5\theta \tag{11}$$

as we find in annex E of the book edited by Wertz [6], in which the Euler–Rodrigues parameters were still called *Euler symmetric parameters*. By applying Eq. (2), we find

$$4ds^2 = d\phi^2 + d\theta^2 + d\psi^2 + 2\cos\theta d\phi d\psi$$

This means that the local differential line-element components of Euler angles are not orthogonal, except for $\theta = \pm\pi/2$. If we consider the gimbal angles (asymmetric Euler angles) ϕ, θ, and ψ consecutively around the coordinate axes x_3, x_2, and x_1, we start from

$$\begin{aligned} x_1 &= \sin\psi/2\cos\theta/2\cos\phi/2 + \cos\psi/2\sin\theta/2\sin\phi/2 \\ x_2 &= \cos\psi/2\sin\theta/2\cos\phi/2 - \sin\psi/2\cos\theta/2\sin\phi/2 \\ x_3 &= \sin\psi/2\sin\theta/2\cos\phi/2 + \cos\psi/2\cos\theta/2\sin\phi/2 \\ x_4 &= \cos\psi/2\cos\theta/2\cos\phi/2 - \sin\psi/2\sin\theta/2\sin\phi/2 \end{aligned} \tag{12}$$

which we derived by considering the individual rotations to be true rotation quaternions that have been multiplied. For the path length, we then obtain

$$4ds^2 = d\phi^2 + d\theta^2 + d\psi^2 + 2\sin\theta d\phi d\psi$$

The differential component orthogonality here is only reached in an area of measure zero: namely, if θ is an integer multiple of π. Nevertheless, these asymmetrical Euler angle differential components are locally close to orthogonality as long as θ has a modest value. Therefore, they are quite adequate to tackle attitude estimation and control problems of airplanes and satellites, which are subject to a tight control law linked to motion (small roll, pitch, and yaw), in which the conventional Euler angles fail to be useful. This was first discovered and exploited by Bryan almost one century ago, whence the name *Bryan angles* has been proposed in this context [3]. This is thus one type of application among others in which regularization is superfluous.

Next, we consider polar coordinates. This parameterization corresponds to the Euler–Rodrigues components in the following four-vector:

$$x_1 = \cos\alpha\cos\delta\sin 0.5\phi \qquad x_2 = \sin\alpha\cos\delta\sin 0.5\phi$$
$$x_3 = \sin\delta\sin 0.5\phi \qquad x_4 = \cos 0.5\phi \tag{13}$$

where α and δ happen to be the right ascension and declination of the rotation axis direction, and ϕ is the size of the rotation around this axis. For easy comparison with the paper on the loxodromic descent [4], we substitute $\pi/2 - \phi/2$ with γ. The infinitesimal geodetic step length with this angular parameterization then becomes

$$ds^2 = \cos^2\delta\cos^2\gamma d\alpha^2 + \cos^2\gamma d\delta^2 + d\gamma^2 \tag{14}$$

Consequently, the vector with the three orthogonal geodetic differential components is now

$$\mathbf{du}' = |\cos\delta\cos\gamma d\alpha, \cos\gamma d\delta, d\gamma| \tag{15}$$

The unit vector $\mathbf{x}$ in Eq. (13) is the Rodrigues four-vector, implying that the differential correction is first expressed as a function of the correction $\boldsymbol{\Delta}\mathbf{x}$. The 4×3 plug-in matrix C accomplishing the transition to the geodetic correction becomes

$$C(\alpha,\delta,\gamma) = \begin{vmatrix} -\sin\alpha & -\cos\alpha\sin\delta & -\cos\alpha\cos\delta\sin\gamma \\ \cos\alpha & -\sin\alpha\sin\delta & -\sin\alpha\cos\delta\sin\gamma \\ 0 & \cos\delta & -\sin\delta\sin\gamma \\ 0 & 0 & \cos\gamma \end{vmatrix} \tag{16}$$

The present parameterization (13) has a convergence problem with the updates of α and δ, especially because an unfavorable synergy occurs when both $\cos\gamma$ and $\cos\delta$ simultaneously approach zero in absolute value. This situation is worsened by the fact that in a differential correction and automatic control, a small correction in terms of rotation angle ϕ may be combined with a large change in the rotation axis. All experiments that we implemented by defining simple inequalities, similar to those working on S^2, failed. We nevertheless succeeded to force the convergence in the test cases by driving the steepest-descent path along loxodromes [4].

The difficulty just met is not unavoidable, because we can define symmetrical spherical coordinates as follows,

$$x_1 = \cos\mu\cos\tau \quad x_2 = \sin\mu\cos\tau \quad x_3 = \cos\nu\sin\tau \quad x_4 = \sin\nu\sin\tau \tag{17}$$

or as any permutation thereof. The corresponding squared differential path length is then equal to

$$ds^2 = \cos^2\tau d\mu^2 + \sin^2\tau d\nu^2 + d\tau^2 \tag{18}$$

with the differential geodetic component vector

$$\mathbf{du}' = |\cos\tau d\mu, \sin\tau d\nu, d\tau| \tag{19}$$

and the plug-in matrix

$$C(\mu,\nu,\tau) = \begin{vmatrix} -\sin\mu & 0 & -\cos\mu\sin\tau \\ \cos\mu & 0 & -\sin\mu\sin\tau \\ 0 & -\sin\nu & \cos\nu\cos\tau \\ 0 & \cos\nu & \sin\nu\cos\tau \end{vmatrix} \tag{20}$$

Although $\sin\tau$ and $\cos\tau$ can both approach 0 and give rise to a parameter indetermination, this cannot happen simultaneously. On the contrary, when the cosine is getting small, the magnitude of the sine approaches 1 and vice versa. We can thus implement the conditions $|\cos\tau| < \epsilon$ and $|\sin\tau| < \epsilon$ and proceed for both obviously separate cases as on S^2. With respect to the loxodromic descent, this is an important algorithmic simplification. It may be expected that adaptations of this regularization principle will be useful in adaptive Kalman filtering and competing methods for attitude control.

Numerical Verification of the Regularization on S^2

To perform the test for the three-dimensional sphere, we determine a unit vector denoted by $\mathbf{n}$, which has to be estimated by employing five scalar measurements m_i. They are obtained from the following scalar products:

$$\mathbf{n} \cdot \mathbf{s_i} = m_i \quad (i = 1, \ldots, 5)$$

where the first four vectors $\mathbf{s_i}$ are unit vectors and $\mathbf{s}_5 = \mathbf{s}_1 \times \mathbf{s}_4$. All of these vectors are initially defined on the unit sphere such that $\mathbf{n}$ is at right ascension $\alpha_n = 30$ deg and declination $\delta_n = 60$ deg and the first four vectors $\mathbf{s_i}$ are in the x–y plane at the right ascensions 0, 10, 20, and 30 deg for $i = 1$ to 4, respectively. The measurements are corrupted with uncorrelated bias-free Gaussian errors with standard deviation $\sigma = 0.001$. The initial guess for $\mathbf{n}$ starts from the renormalized error-free vector first offset by $|0.05, 0.05, -0.05|'$. The resulting initial direction error is approximately 4.5 deg. The actual test is performed by repeating the estimation trials of a large sample with the same relative measurement geometries but rotated into different absolute orientations, each with the same batch of random errors. Thus, in each sample, all vectors comprising both $\mathbf{n}$ and $\mathbf{s_i}$ are rotated together as a rigid unity, such that $\mathbf{n}$ moves toward the z axis up to the point at which $\mathbf{n}$ and the z axis coincide. The rotation axis of these transformations is chosen to be perpendicular to the z-axis $\mathbf{n}$ plane. In this way, the error-free and the error-corrupted measurements always remain the same. This procedure has been applied to numerous cases (orientations) with $60 \text{ deg} \leq \delta_n \leq 90$ deg, each containing 10,000 trials. The iteration stop criterion of the steepest descent was defined by

$$\sqrt{\cos^2\delta\Delta\alpha^2 + \Delta\delta^2} \le \eta$$

The different tests were compared on the basis of their mean (attitude) ε and root-mean-square (RMSQ) error in degrees. A sample of the test results is displayed in Table 1. They are based on the values $\eta = 10^{-6}$ deg and $\epsilon \le 10^{-12}$, as suggested earlier. The maximum iteration means the largest iteration number observed in a single trial of the sample. By performing the computation with 64-bit floating-point arithmetic, we could identify a difference of 1 at the 10th significant digit of the mean value ε, but no difference of the RMSQ values at that level. Widening the value of ϵ to 10^{-10} and keeping $\eta = 10^{-6}$ deg affected the results for $89.60\,\mathrm{deg} \le \delta_N$, in which one or two of the 10,000 trials of the sample did not achieve convergence after 20 iterations and thereby ε slightly increased beyond the fifth decimal. On the contrary, setting $\eta = 10^{-7}$ deg and $\epsilon \le 10^{-13}$ did not modify the results of Table 1, except that the mean number of iterations systematically increased by approximately 0.3. The maximum number of iterations both increased and decreased at random by 2 at most.

Numerical Verification of the Regularization on S^3

Algorithmic Considerations

Numerical work with angles on S^3 normally implies a choice to cope with the multiple ambiguities that exist when any unit four-vector is translated in the angles of any *particular* spherical coordinate representation. Within the constraints resulting from such a choice, we further have to make sure to stay in consistent definition intervals when implementing the updates of steepest descent. Even if this may seem trivial at first glance, not all simple recipes lead to an always-correctly-working algorithm. Therefore, we briefly explain the (not unique) way we have resolved these practical problems in the case of the spherical coordinates given in Eq. (17).

The τ coordinate will always be selected to be in the first quadrant, or $0 \le \tau \le \pi/2$. Hence, this angle can be computed by

$$\tau = \arctan\frac{+\sqrt{x_3^2 + x_4^2}}{+\sqrt{x_1^2 + x_2^2}}$$

This constraint still allows us to reach any sign combination of the unit four-vector by selecting μ and ν to be adequately defined in a 2π interval. Recalling that the argument function $\arg(u, v)$ with $u^2 + v^2 = 1$, known from basic complex function theory, computes the angle $0 \le \omega < 2\pi$ for $u = \sin\omega$ and $v = \cos\omega$, we select

$$\mu = \arg(x_2, x_1) \qquad \nu = \arg(x_4, x_3)$$

because both $\cos\tau$ and $\sin\tau$ are positive or zero by definition. Depending on the specific computer implementation of the argument function, it may be necessary to modify the result to obtain $-\pi \le (\mu, \nu) \le +\pi$.

During steepest descent, we have to take care that the differential increment $\Delta u_3 = \Delta\tau$ does not move the update $\tau = \tau_0 + \Delta\tau$ outside the definition interval $[0, \pi/2]$. Therefore, a restraint factor f is included, starting with $f = 1$, and repeatedly subjected to quadrisection (resulting in powers of 0.25) until $\tau = \tau_0 + f\Delta\tau$ is inside the prescribed interval. Numerical experience shows that a slower reduction speed such as $f = 1, 0.9, 0.8, \ldots$ may, especially in the first iterations, bring us very close to either zero or $\pi/2$ right from the start. This may cause a lockup in a secondary minimum. It can further happen that one has reached $\tau_0 = \pi/2$ within 10^{-15} rad, for instance. Any positive increment $\Delta\tau$ will then lead to very long loops in trying to find an adequate value of f. This could be avoided by allowing five consecutive reductions before deciding not to update τ_0. All examples given in Table 2 only use quadrisection without limiting the little f loop.

At this stage, the updated values of $\sin\tau$ and $\cos\tau$ are available. The angle μ can consequently be incremented. If $|\cos\tau| < 10^{-12}$, we set $\mu = 0$, disregarding the value of $\Delta e_1 = \cos\tau\Delta\mu$, and proceed with the update of ν. If not, we verify whether

$$|\Delta\mu| = |\Delta e_1/\cos\tau| \le 10\,\mathrm{deg}$$

Should this not be the case, we reduce $\Delta\mu$ to 10 deg in absolute value. Next we get the update $\mu = \mu_0 + f\Delta\mu$. As a further precaution, we take μ modulo 2π. Finally, we replace μ by $2\pi + \mu$ if $\mu < -\pi$ and by $\mu - 2\pi$ if $\pi < \mu$ to get the full precision of the intrinsic cosine and sine computer functions. The handling of ν is the same, but now $\cos\tau$ is to be replaced by $\sin\tau$.

The iterations are stopped if

$$\sqrt{\mathbf{\Delta u}', \mathbf{\Delta u}} = \eta \le 10^{-6}\ \mathrm{deg}$$

or if $|\eta_i - \eta_{i+1}| \le 10^{-6}$ deg, where i and $i + 1$ refer to successive iterations. Alternatively, reaching the iteration limit of 20 (30 in a few cases) stops the iteration process as well. Nonetheless, being stopped by the iteration limit is not considered to be an error, as will be seen hereafter. Slight oscillations of the η value up and down are also accepted. We have not implemented an iteration failure test, which has to be applied to real-life data to avoid the acceptance of fully inconsistent measurement and reference input. To compensate this omission, we have tabulated the error frequencies of ε defined in the next section in 120 bins of $0.008\overline{3}$ deg wide to spot any outlier that may get hidden in the mean error $\overline{\varepsilon}$ over 10,000 trials.

Description of the Test Estimation

For the numerical verifications, we reuse the simple attitude measurement scenario already presented in the Note about the

Table 1 Performance within 10 deg of the original singularity

δN, deg	Mean iteration	Maximum iteration	Mean ε	RMSQ
80.00	6.67	8	0.1234527748	0.1493497130
89.00	7.05	15	0.1234527747	0.1493497130
89.60	7.49	18	0.1234527748	0.1493497130
89.99	7.09	15	0.1234527747	0.1493497130
90.00	7.16	14	0.1234527747	0.1493497130

Table 2 Test-case parameters

Case	x_1	x_2	x_3	x_4	Φ, deg	Θ, deg	Ψ, deg
1	0.0	0.0	0.0	1.0	0.0	0.0	0.0
2	0.017452	0.0	0.0	−0.99985	−2.0	0.0	0.0
3	-0.870×10^{-3}	-0.437×10^{-3}	0.209×10^{-2}	0.99999	0.09977	0.04985	−0.23927
4	0.0	0.0	0.67984	−0.73336	0.0	0.0	−85.662
5	0.96786	0.25148	0.0	0.0	−180.00	0.0	−29.130
6	-0.454×10^{-2}	-0.118×10^{-2}	0.99999	-0.170×10^{-7}	0.13530	−0.52070	−179.99
7	0.0	0.67984	0.0	−0.73336	0.0	−85.662	0.0

Table 3 Enhanced-accuracy spot on the singularity

Case	Mean, deg	RMSQ, deg	Sample size	$(\text{Mean})_S$ deg	$(\text{RMSQ})_S$ deg
1	0.079392	0.088633	13	0.042998	0.050136
4	0.079757	0.088944	812	0.051403	0.057522
5	0.079651	0.088897	896	0.049253	0.055025

loxodromic descent [4]. It comprises two baseline unit vectors **b** laying in different planes intersecting on the x axis. These baselines measure the projection of up to six reference directions, which one can imagine to correspond to Global Positioning System satellite directions that are not all simultaneously visible in the different hemispheres. The baselines are measuring mainly in the hemisphere toward the $+z$ direction. Right ascension and declination expressed in degrees are employed to define all of these directions. For the baselines, one has

$$\mathbf{b}_1(90,-30) = |\cos 90 \cos 30,\quad \sin 90 \cos 30,\quad -\sin 30|$$

and $\mathbf{b}_2(270,-30)$. Similarly, the six observed reference unit vectors are arbitrarily selected to be

$$\mathbf{a}_1(47,-10) \qquad \mathbf{a}_2(65,20) \qquad \mathbf{a}_3(181,63)$$
$$\mathbf{a}_4(142,41) \qquad \mathbf{a}_5(222,-3) \qquad \mathbf{a}_6(313,57)$$

All of the measurements with baseline visibility at 2 deg from grazing incidence are employed (in fact, 7 in the present case). Moreover, all measurements are subjected to bias-free uncorrelated normal distributed random errors with a standard deviation of $\sigma = 0.001$. Let $R_x(a)$ represent a rotation of a deg around the x axis. We report hereafter the results of seven test rotations $R_0 = R_x(\psi)R_z(\theta)R_x(\phi)$ that are implemented to keep the mutual orientation of baselines and reference directions unchanged but described in rotated reference frames, always resulting in the same measurements $\mathbf{b}'_\mathbf{i} \cdot \mathbf{a}_\mathbf{j} = m_{ij}$, by setting

$$\mathbf{b}'_\mathbf{i} R_0(R'_0\mathbf{a}_\mathbf{j}) = m_{ij} \qquad i = 1,2 \quad \text{and} \quad j = 1,\ldots,6 \tag{21}$$

The value of the initial rotation R_{init} at the start of the iteration has always been selected to be $R_{\text{init}} = R_z(3)R_y(3)R_x(3)R_0$, for which the error rotation angle derived from the trace of $R_e = R_z(3)R_y(3)R_x(3)$ corresponds to approximately 7 deg. If R is the estimated rotation and $R = R_eR_0$, then in the same way, the error rotation ε is extracted from trace $(R_e) = 1 + 2\cos\varepsilon$.

Test Results

The specific examples displayed in Table 2 have been selected to cover a representative sample of Rodrigues four-vectors, which may hide a problem. To characterize these cases, Table 2 shows the values of Euler–Rodrigues parameters with the symmetric Euler angles belonging to R_0. These examples contain a few cases at the location of the original singularity and others that have been chosen to verify the correct algorithmic treatment of the trigonometric constraints applicable to successive iteration steps. The critical appreciation of the estimation accuracy relies on the error rotation angle ε. Thereby, it appears that the mean estimation errors $\overline{\varepsilon}$ and the corresponding RMSQ error are each identical up to ten decimal places in all examples in which τ is neither equal to $\pm\pi/2$, 0.0, or $\pm\pi$. More precisely, we always obtain $\overline{\varepsilon} = 0.0798404047$ and $\overline{(\varepsilon^2)^{1/2}} = 0.0889876824$, even if there is some variation in the number of iterations required. These iterations are limited to 20, and reaching this limit only occurred in less than 5 of the 10,000 trials of the sample.

The situation is different in cases 1 and 4 (in which the error-free value of τ is $\pi/2$) and in case 5 (in which $\tau = 0$), corresponding to original singularities of the spherical representation (17). In the latter three cases, detailed in Table 3, we had to increase the maximum allowed iteration limit to 30 to obtain mean and RMSQ errors that would not decrease any more by applying further iterations. The overall results for $\overline{\varepsilon}$ and $\overline{(\varepsilon^2)^{1/2}}$ are slightly better than in the other test cases, starting from the third significant digit onward. We also computed a separate $(\text{mean})_S$ and $(\text{RMSQ})_S$ errors for the estimation results obtained on the basis of the maximum number of iterations, for which the sample size is added in the fourth column of Table 3. The latter results are substantially better than the overall mean values (which also comprise the results just singled out) shown in columns 2 and 3. Moreover, for cases 4 and 5, the observed accuracy improvement of almost 50% applies to some 8.5% of the trials, and the corresponding mean and RMSQ values are mutually comparable in both cases. We may thus assume that all this is not incidental and has a theoretical background that has not yet been investigated. The highest errors inside each sample always occurred in the same bins (which are 0.008333 deg wide, as mentioned earlier), with the same densities in each sample, and amounted to seven estimates (of 10,000) with an error between 0.5 and 0.56 deg.

Conclusions

It has been shown on the basis of differential geometry that the occurrence of algebraic singularities can completely be avoided in transitions from a Cartesian direction specification, in both dimensions three and four, to particular minimum parameter specifications in a nonlinear estimation problem. This was made possible by implementing the linearization of conventional polar coordinates for localizing points on the three-dimensional unit sphere and symmetrical spherical coordinates on the unit sphere of dimension four and considering geodetic increments instead of coordinate increments. The numerical tests, with 64-bit floating-point arithmetic, further demonstrated that one could approach the original singularity, consisting of either a sine or cosine becoming zero, up to a value as small as 10^{-12} before being obliged to set the relevant undetermined coordinate, typical for the singularity, to an ad hoc value. The mean value of the required modest number of iterations was hardly affected by the proximity of the original singularity, and the test estimates were equal in the mean up to 10 decimal places. In dimension four, the required number of iterations right at the location of the singularity went up to 30 in, at most, 10% of the trials, but in these particular cases, the mean estimation precision was almost doubled.

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